Introduction to Probability Theory
9th week: Multidimensional continuous random variables.

Let us recall that by a 2-**dimensional random variable** we mean a (measurable) function

$$(X,Y) : \Omega \to \mathbb{R}^2 ,$$

where Ω is the sample space of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The **cumulative distribution function (CDF)** of (X,Y) is a function $F_{X,Y} : \mathbb{R}^2 \to [0,1]$ defined as

$$F_{X,Y}(x,y) = \mathbb{P}\left(X \leqslant x, Y \leqslant y\right) .$$

Definition. *If the CDF $F_{X,Y}$ of $(X,Y) : \Omega \to \mathbb{R}^2$ can be written as*

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t) ds dt$$

for some non-negative function $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$, then we say that (X,Y) is **(absolutely) continuous***, and $f_{X,Y}$ is the* **density** *of (X,Y).*

In such a case

$$f_{X,Y}(s,t) = \frac{\partial^2 F(x,y)}{\partial x \partial y}\Big|_{x=s, t=y},$$

for (nearly) all values of (s,t), and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(s,t) ds dt = 1 .$$

If (X,Y) is (absolutely) continuous, then the CDF and the density of X are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,t) dt ,$$

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) ,$$

while for Y analogous formulae are

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(s,y) ds$$

and

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) .$$

If $f_{X,Y}$ is the density of (X,Y), then for each (measurable) function $g : \mathbb{R}^2 \to \mathbb{R}$ we have

$$\mathbb{E}g(X,Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(s,t) f_{X,Y}(s,t) dt ds ,$$

so, for instance,

$$\mathbb{E}X^2Y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s^2 t f_{X,Y}(s,t) dt .$$

Definition. *Random variables X and Y are* **independent***, if for every x and y,*

$$F_{X,Y}(x,y) = F_X(x) F_Y(y) .$$

Moreover, if $f_{X,Y}$ is the density of a continuous random variable (X,Y), then X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) ,$$

for (nearly) all values of x i y.