Introduction to Probability Theory
8th week: Continuous random variables.

Definition. *A random variable X is* **continuous** *if its CDF $F_X : \mathbb{R} \to \mathbb{R}$, defined as*

$$FX = \mathbb{P}(X \leqslant x),$$

is continuous for all $x \in \mathbb{R}$.

Equivalently, X is continuous if it contains no atoms, i.e. $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$.

Remark: Most of random variables we will study are either discrete, or continuous, but please keep in mind that there are random variables which are neither discrete, nor continuous.

For a wide class of continuous random variables X (so called **absolutely continuous** random variables) which include almost every continuous random variable we shall study in this course, there exists a function $f : \mathbb{R} \to \mathbb{R}$, called the **density** of X, such that for every $x \in \mathbb{R}$ we have

$$F_X(x) = \mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} f(t)dt.$$

Then,

$$f(x) = F_X'(x).$$

Remark: The CDF F_X is uniquely defined for every random variable X. The density function f, even if it exists, always appear under integral and so it is definied only up to a set of measure zero. Hence, we can change its value in a few points and this will not affect any of the statements on it. However, it is convenient to assume that, say, it is always non-negative (see the statement below).

Properties of the density function:

(i) $f(x) \geqslant 0$, for all $x \in \mathbb{R}$;

(ii) $\int_{-\infty}^{\infty} f(t)dt = 1$.

Homework assignment for Quiz 6, April 25th

Problem 1. Let

$$F(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ x^3/8 & \text{for } 0 \leqslant x \leqslant 2 \\ 1 & \text{for} \quad x \geqslant 2. \end{cases}$$

be the CDF of X. Find the density of X.

Problem 2. The density of a random variable X is given by the following formula:

$$f(x) = \begin{cases} 1/3 & \text{for } x \in [0,3] \\ 0 & \text{for } x \notin [0,3]. \end{cases}$$

Find the CDF of X.

Problem 3. Mr Smith each Monday comes to a bus station in a random moment between 7:00 and 8:00AM. He does not realize that the bus arrives at his bus stop precisely at 7:10, 7:40 i 8:05. Let X denote the time Mr Smith is waiting for the bus, measured in minutes (and fractions of minutes, since the time is, of course, continuous). Find the CDF and the density of X.

Problem 4. We break a stick of length 1 in a random place. Let X denote the length of a shorter of the two parts obtained. Find the CDF and the density of X.

Problem 5. We shoot at a target which is a square of length 1. With each shot we win $2k$, where k is the distance from the place we hit to the nearest side of the square. By X we denote the win in this game (so, for instance, if we hit the center of the square we win 1). Find the CDF and the density of X.