Introduction to Probability Theory
5th week: Random Variables, their distribution and expectation.

Definition. *By a* **random variable** *we mean a (measurable) function $X : \Omega \to \mathbb{R}$, where Ω is the sample space of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

For $A \subseteq \mathbb{R}$ we write $\mathbb{P}(X \in A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$.

Definition. *A random variable X is* **discrete**, *if for some countable set $A = \{a_1, a_2, \ldots\}$ (whose elements we call* **atoms***) we have*

$$\mathbb{P}(X \in A) = \sum_{a_i \in A} \mathbb{P}(X = a_i) = 1.$$

For a discrete random variable X the probabilities $p_i = \mathbb{P}(X = a_i)$ determine the **probability mass function (pmf)** which describes the **(probability) distribution of** X.

Definition. *The* **cumulative distribution function (CDF)** *of a random variable X is a function $F_X : \mathbb{R} \to \mathbb{R}$ defined as*

$$F_X(x) = \mathbb{P}(X \leqslant x).$$

In particular, for a discrete random variable X with atoms $A = \{a_1, a_2, \ldots\}$ we have

$$F_X(x) = \sum_{a_i \in A, a_i \leqslant x} \mathbb{P}(X = a_i).$$

Definition. *The* **expectation** *(or the* **mean***) of a random variable X is a real number defined as*

$$\mathbb{E}X = \sum_{a_i \in A} a_i \mathbb{P}(X = a_i).$$

If the series on the right hand side is not absolutely convergent then we say that the expectation of X does not exist.

Problems we shall do in class on March 28th

Problem 1. The pmf of a random variable X is given in the following table

k	1	2	3	4	5
$\mathbb{P}(X = k)$	$\frac{2}{15}$	$\frac{5}{15}$	$\frac{4}{15}$	?	$\frac{1}{15}$

Put in the table the missing number and compute (and draw) the CDF of X. Find $\mathbb{P}(2.5 \leqslant X \leqslant \pi)$, $\mathbb{P}(X = \pi)$, $\mathbb{P}(X \geqslant \pi)$, and compute $\mathbb{E}X$.

Problem 2. The CDF of X is given by

$$F(x) = \begin{cases} 0 & \text{for } x < a; \\ \frac{1}{6} & \text{for } a \leqslant x < 0; \\ \frac{1}{2} & \text{for } 0 \leqslant x < 1; \\ \frac{5}{6} & \text{for } 1 \leqslant x < b; \\ 1 & \text{for } x \geqslant b. \end{cases}$$

Find the pmf of X and compute $\mathbb{E}X$.

Problem 3. A sniper has three bullets and keeps shooting at a target until he either hits a target or runs out of the bullets. The probability of hitting the target in a single shot is 0.8. Let X denote the number of bullets shot by a sniper.

a) Describe the sample space Ω which is the domain of random variable X.
b) List all elements of Ω which belong to the events $\{\omega \in \Omega : X(\omega) = 3\} = \{X = 3\}$ and $\{\omega \in \Omega : X(\omega) \leqslant 2\} = \{X \leqslant 2\}$ and find the probability of these events.
c) Find the pmf and the CFM of X.
d) Find $\mathbb{E}X$.

Problem 4. An urn contains 3 white and 2 black balls. We select from the urn 3 different balls. Let X denote the number of black balls among three balls we choose.

a) What is the number of elements in Ω?
b) Find the pmf of X and $\mathbb{E}X$.

Problem 5. From the set $\{1, 2, 3, 4, 5\}$ we choose three different numbers $x < y < z$. Let Y denote the one which lie in the middle.

a) Find the domain Ω of Y.
b) List all the elements of the events $\{\omega \in \Omega : Y \leqslant \sqrt{5}\} = \{Y \leqslant \sqrt{5}\}$ oraz $\{\omega \in \Omega : Y(\omega) > 2\} = \{Y > 2\}$ and find the probability of these events..
c) Find the pmf of Y.
d) Draw the graph of the CDF of Y.
e) Compute $\mathbb{E}X$.

Homework assignment for Quiz 4, March 28th

Problem 1. Find the pmf and $\mathbb{E}X$ of a random variable X with the CDF given by

$$F(x) = \begin{cases} 0 \text{ dla } x < -5; \\ \frac{1}{6} \text{ dla } -5 \leqslant x < 1; \\ \frac{1}{3} \text{ dla } 1 \leqslant x < 4; \\ \frac{1}{2} \text{ dla } 4 \leqslant x < 10; \\ 1 \text{ dla } x \geqslant 10. \end{cases}$$

Problem 2. We flip a coin and, at the same time, toss a die. We win 4\$ when we get Head and One, we win 2\$ when we get Tail and an even number, and in all other cases we lose 3\$ (i.e. we win -3\$). Let X be the random variable which is the win in this game. Describe the sample space Ω which is the domain of X, find the pmf and the CDF of X, and compute $\mathbb{E}X$.

Problem 3. We choose two different balls from an urn containing 6 white, 4 black, and 2 red balls. We are paid 1\$ for each black ball, lose 1\$ for each white ball, and we neither win nor lose if we pick a red ball. By X we denote our total gain. Find the domain Ω of X and list all elements of the events $\{\omega \in \Omega : X(\omega) = 0\} = \{X = 0\}$ oraz $\{\omega \in \Omega : X(\omega) \leqslant -1\} = \{X \leqslant -1\}$. Find the pmf and the CDF of X, and $\mathbb{E}X$.

Problem 4. A car dealer is going to meet two clients. He estimates that the first client will buy a car with probability 0.3, for the second client the probability is 0.6. Moreover, for each of the clients, the probability that he buys a basic version of a car for 20 000\$ and that he buys its high-end version for 30 000\$ is roughly the same. Let X denote the total amount of money the dealer will get from these two clients. Find the pmf of X and $\mathbb{E}X$.

Problem 5. From the set $\{1, 2, \ldots, 200\}$ we select two different numbers. By X we denote the larger of these two numbers. Find the pmf of X and compute $\mathbb{E}X$.