Introduction to Probability Theory
4th week: The chain rule. The Law of Total Probability. Bayes' Theorem.

Theorem (Chain rule). *If for a sequence of events A_1, A_2, ..., A_n we have $\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0$, then*

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1) \cdot \ldots \cdot \mathbb{P}(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}).$$

Theorem. *Let us suppose that Ω is partitioned into events $B_1, B_2, \ldots, B_k \in \mathcal{F}$, each of positive probability. Then for every event $A \in \mathcal{F}$ we have*

$$\mathbb{P}(A) = \sum_{i=1}^{k} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)\,,$$

and for every $j = 1, 2, \ldots, k$,

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\,\mathbb{P}(B_j)}{\sum_{i=1}^{k} \mathbb{P}(A|B_i)\,\mathbb{P}(B_i)}\,.$$

The first equation in the above theorem is called **the Law of Total Probability**, while the second one is **Bayes' Theorem**.

Problems we shall do in class on March 21th

Problem 1. In a car factory there are three production lines L_1, L_2, and L_3 which produce 25%, 35%, and 40% of all cars respectively. 5% of all cars produced by the first line L_1 have problems with brakes, for the lines L_2 these problems appear in 4% of cars, and for L_3 this fraction is just 2%.

(a) What is the probability that a randomly selected car from this factory will have problems with brakes?

b) Suppose that a randomly selected cars from this factory has problems with brakes. What is the probability that it came from the line L_1?

Problem 2. We send a sequence of bits through a non-reliable channel. It replaces 0 with 1 with probability 0.2, and change 1 to 0 with probability 0.1. We know also that 25% of bits of the sequence are zeros. Suppose that we receive 0. What is the probability that this bit has not been corrupted by the channel?

Problem 3. In the first urn there are two white and two black balls; the second urn contains one white and two black balls. We select randomly two balls from the first urn and put them into the second urn; then we select from it one ball. Find the probability, that we put in the second urn two white balls provided we know that the ball we chose in the second round is (a) white, (b) black.

Problem 4. A box contains four white and two black balls. We pick randomly one ball, color it red, and put it back in the box. Then, in the second round, we select randomly one ball.

(a) Find the probability that the ball chosen in the second round is white.

(b) Suppose that the ball selected in the second round is white. What is the probability that the ball chosen in the first round was white as well?

(c) Suppose that the ball selected in the second round is red. What is the probability that the ball chosen in the first round was white? Is this result surprising?

Problem 5. Each morning Bob and Kate go to work by bus. They arrive at the bus stop independently of each other, at a random moment between 8:00 and 8:20 pm. Their bus arrives at the stop at a random moment between 8:00 and 8:10. One day Bob comes to the bus stop at 8:05 and realizes that Kate is not there. What is the probability that he missed the bus?

Homework assignment for Quiz 3, March 21st

Problem 1. An urn contains 3 white balls, 2 black balls, and 1 red ball. We select from it three balls, one by one, but we do **not** return the chosen ball back to the urn after each round. Compute the probability that we will choose a white ball in the first round, a black ball in the second round, and a white one in the third round.

Problem 2. Three groups of students G_1, G_2 i G_3, of 20, 25 i 20 students respectively, prepare to the quiz. In each of them the number of students who have worked out all the problems from the last homework assignments is 8, 10, and 10 respectively. All groups write quiz at the same time and then the instructor picks up a random quiz and grade it. What is the probability that the quiz was written by somebody from the group G_1, if it turns out that it got the maximum number of points? Of course, we assume that those studends who worked all problems (and only them) got the maximum number of points from the quiz.

Problem 3. About 5% of men and 0,25% of women are colour-blind. Let us choose randomly a person P (we assume that a chance to pick up a man and a woman are the same).

(a) What is the probability that P is colour-blind?
(b) What is the probability that P is a man provided we know P is colour-blind?

Problem 4. Each of three urns contains 4 balls. In the ith urn, $i = 1, 2, 3$, there is i white balls and $4 - i$ black balls. We pick up randomly one urn (each with probability 1/3) and select from it two different balls. What is the probability, that we choose balls of different colours? What is the probability that we selected them from the first urn provided we know that they are of different colours?

Problem 5. An urn contains 4 white balls and two black balls. We select one ball from the urn, repaint it into "the other" colour (i.e. we colour a white ball black and black one with white) and put it back into the urn. What is the probability that if we select a ball from urn now we pick up a white ball?

Problem 6. 60% for all e-mails received by John are spam messages. The word *winner* appears in 10% of all spam messages and just in 1% other messages. Compute the probability that the next message containing the word *winner* will be a spam message. Find the probability that the next message without the word *winner* will be not a spam message.