Introduction to Probability Theory
3rd week: Probability Axioms. Conditional Probability

Definition. *A* **probability space** *is a triple* $(\Omega, \mathcal{F}, \mathbb{P})$*, where* Ω *is a set called a* **sample space***,* $\mathcal{F}$ *is a special family (*σ*-field) of subsets of* Ω *called* **events***, and* $\mathbb{P} : \mathcal{F} \to [0,1]$ *is a* **probability function** *such that*

(i) $0 \leqslant \mathbb{P}(A) \leqslant 1$ *for every* $A \subseteq \Omega$*,*
(ii) $\mathbb{P}(\Omega) = 1$*,*
(iii) *for a sequence of pairwise disjoint events* $A_1, A_2, \ldots \in \mathcal{F}$*, we have* $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$*.*

Definition. *If* $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ *and* $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ *are probability spaces by their* **product space** *we mean the space* $(\Omega, \mathcal{F}, \mathbb{P})$ *where* $\Omega = \Omega_1 \times \Omega_2$*,* $\mathcal{F}$ *contains all the events* $A_1 \times A_2$*, where* $A_1 \in \mathcal{F}_1$ *and* $A_2 \in \mathcal{F}_2$*, and for such a pair of events we have*

$$\mathbb{P}(A_1 \times A_2) = \mathbb{P}_1(A_1) \cdot \mathbb{P}_2(A_2).$$

In a similar way we define the product of a sequence (finite of infinite) of probability spaces.

Examples:

Bernoulli scheme. Let $p \in (0,1)$ and let n be a natural number. We repeat n times a random experiment in which the probability of a success is p (and the probability of a failure is $1-p$). If by ρ_k we denote the probability that precisely k times we succeed, then

$$\rho_k = \binom{n}{k} p^k (1-p)^k \quad \text{for} \quad k = 0, 1, \ldots, n.$$

Geometric distribution. Suppose that we repeat a random experiment in which the probability of a success is p. Let σ_k denote the number of trials we need to get the first success. Then

$$\sigma_k = (1-p)^{k-1} p \quad \text{for} \quad k = 1, 2, \ldots.$$

Definition. *If* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* $B \in \mathcal{F}$ *is an event* **such that** $\mathbb{P}(B) > 0$*, then we can construct a new probability space* $(B, \mathcal{F}_B, \mathbb{P}_B)$ *setting*

$$\mathcal{F}_B = \{F \cap B : F \in \mathcal{F}\}$$

and for each $A \in \mathcal{F}_B$

$$\mathbb{P}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

The probability $\mathbb{P}_B(A)$ *we typically denote as* $\mathbb{P}(A|B)$ *and call it* **the probability of** A **under the condition** B*, or* **the conditional probability of** A **given** B**.**

Remark. Note that for two independent events A, B we have

$$\mathbb{P}(A|B) = \mathbb{P}(A) \quad \text{and} \quad \mathbb{P}(B|A) = \mathbb{P}(B),$$

provided $\mathbb{P}(A), \mathbb{P}(B) > 0$ so the left hand sides are well defined.

Problems we shall do in class on March 14th

Problem 1. Prove that for a sequence of events $A_1, A_2, \ldots$ we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Problem 2. Let $A_1, A_2, \ldots$ be events such that for every $i = 1, 2, \ldots$, we have $\mathbb{P}(A_i') \leqslant 3^{-i}$. Prove that

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) \geqslant 1/2.$$

Is it true that if $\mathbb{P}(A_i') = 3^{-i}$, for every $i = 1, 2, \ldots$, then we have

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1/2\,?$$

Problem 3. We flip a coin until we get a head. Let E denote the events that we flip it even number of times and $A_{\geqslant k}$ the event, that we flip it at least k times.

(a) Compute $\mathbb{P}(E)$, $\mathbb{P}(E|A_{\geqslant 3})$, and $\mathbb{P}(E|A_{\geqslant 4})$.
(b) Are events E and $A_{\geqslant 3}$ independent?
(c) Are events E and $A_{\geqslant 4}$ independent?

Problem 4. We toss a die four times. Let A be an event that precisely twice we get an even number and B denote the event that precisely one time we get six. Compute $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$. Are events A and B independent?

Problem 5. An urn contains 3 red balls, 2 blue balls and 1 green ball. We select from it two different balls, return them to the urn, and repeat the process again.

(a) Find the probability that in the first n rounds, exactly k times we choose two balls of different colours.
(b) Find the probability that we choose two balls of different colours for the first time in the nth round.
(c) We continue the process until we choose two balls of different colours twice (not necessarily in the consecutive turns). Find the probability that there will be precisely n rounds.
(d) We continue the process until we choose two balls of different colours in two consecutive turns. Find the probability that there will be precisely n rounds.

Homework assignment
(you will be given a similar problem on Quiz 2, March 14th)

Problem 1. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, $\mathbb{P}(\{\omega_1\}) = \frac{1}{8}$, $\mathbb{P}(\{\omega_2\}) = \mathbb{P}(\{\omega_3\}) = \mathbb{P}(\{\omega_4\}) = \frac{3}{16}$, and $\mathbb{P}(\{\omega_5\}) = \frac{5}{16}$. Moreover, let $A = \{\omega_1, \omega_2, \omega_3\}$, $B = \{\omega_1, \omega_2, \omega_4\}$, and $C = \{\omega_1, \omega_3, \omega_4\}$. Verify that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$, but show that events A, B, C are not mutually independent.

Problem 2. On an exam a student has to answer two questions: the first one on probability, the other on data bases. The probability that he can (correctly) answer the first question is 0.6, the probability of answering (correctly) the second one is 0.8 and the probability that he can (correctly) answer both of them is 0.5. Find the probability that

(a) he can answer the first question but cannot answer the second one,
(b) he can answer both questions,
(c) he can answer just one of the two questions,
(d) he can answer neither of the two questions.

Find the conditional probability that he can answer the first one under the condition that he can answer the second one.

Problem 3. Find the probability that tossing a die five times exactly three times we get at least five.

Problem 4. An urn contains three red balls, two blue balls, and one green ball. We select randomly two balls. What is the conditional probability that we choose balls of different colour under the condition that at least one ball is red.

Problem 5. From an urn which contains three red balls, two blue balls, and one green ball we select randomly two balls. Then we return them to the urn and repeat the same choice ten times. Compute the probability that precisely four times we choose two blue balls.

Problem 6. We repeatedly toss a die until we get six. Let $D_{\leqslant k}$ be the event that we will toss a die at most k times.

(a) Find the probability of $D_{\leqslant 3}$.
(b) Find $\mathbb{P}(D_{\leqslant 3}|D_{\leqslant 2})$ and $\mathbb{P}(D_{\leqslant 2}|D_{\leqslant 3})$.

(c) Find the conditional probability that we will toss a die precisely three times given that we get at least one five in the first two tosses.