Introduction to Probability Theory
2nd week: Geometric Probability. Independence.

Definition. *We say that events* $\{A_i : i \in I\}$ *are* **mutually independent**, *if for each (finite)* $J \subseteq I$ *we have*

$$\mathbb{P}\left(\bigcap_{j\in J} A_j\right) = \prod_{j\in J} \mathbb{P}(A_j)$$

Thus, for instance, we say that two events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

On the other hand, three events A, B, and C are mutually independent if **each** of the four following equalities holds

$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$,
$\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$,
$\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C)$,
$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$.

Problems we shall do in class on March 7th

Problem 1. We toss three dice. Let A denote the event that we get an even number on the first die, B be the event that we get an odd number on the second die, and and C be the event that the results on all three dice are different.

(a) Are events A and B independent?
(b) Are events A and C independent?
(c) Are events A, B, and C mutually independent?

Problem 2. We pick two random points in the interval of length 10. Find the probability of the event D_3 that the distance between them is 3, and the event $D_{>3}$ that it is larger than 3.

Problem 3. In the interval $[0, 1]$ we select randomly two points, dividing it into three smaller intervals of length a, b, and c, respectively. What is the probability that there exists a triangle with sides of length a, b, and c?

Problem 4. On an exam a student has to answer two questions: the first on probability, the other on data bases. The probability that he can (correctly) answer the first question is 0,6, the probability of answering (correctly) the second one is 0,8. Let us make a rather unrealistic assumption that the events that he answers correctly either of the questions are independent. Find the probability that

(a) he can answer the first question but cannot answer the second one,
(b) he can answer both questions,
(c) he can answer just one of the two questions,
(d) he can answer neither of the two questions.

Problem 5. We choose randomly a number x from the set $\{1, \ldots, 1800\}$. By A_k we denote the event that k divides x.

(a) Are events A_5 i A_6 independent?
(b) Are events A_6 i A_9 independent?

Homework assignment
(you will be given a similar problem on Quiz 1, March 7th)

Problem 1. We choose randomly on point a in a unit square. Compute the probability that

(a) a belongs to one of the diagonals of the square.
(b) a belongs to one of the sides of the square?
(c) a lies within distance $1/2$ from the center of the square?

Problem 2. We pick randomly two points x, y, in the interval $[0, 8]$. Find the probability that the center c of the interval $[x, y]$ belongs to $[2, 4]$.

Hint *Note that* $c = (x + y)/2$.

Problem 3. We flip a coin four times. Let A be the event that in the first flip we get tail, and by B we denote the event that in the four flips we get tail precisely two times. Are A and B independent?

Problem 4. We randomly select two cards from the standard deck of 52 cards. By A we denote the event that both of them are spades, by B that we get a king and a queen, and by C that no cards of value from 2,3,4,..., 9 are among the chosen two.

(a) Are events A and B independent?

(b) Are events A, B, and C mutually independent?

Problem 5. A network fails if either a server K_1 fails, or both servers K_2 and K_3 stop working at the same time. The servers K_1, K_2, K_3 fails happens independently with probabilities 0.3, 0.2, and 0.1, respectively. Calculate the probability that the network will fail.

Problem 6. Each among 100 computer scientists working in some corporation arrives at work at a random moment between 8:00 AM and 9:00 AM. Find the probability that at 8:15 AM precisely 25 of them are at work.