Introduction to Probability Theory
14th week: Two-Person Zero-Sum Game.

Definition. *By a* **two-person zero-sum game** *(or, more precisely, by a* **normal** *or* **strategic form** *of such game) we mean a triple (X, Y, A), where $X = \{x_1, \dots, x_n\}$ is the set of pure strategies of the first player, $Y = \{y_1, \dots, y_m\}$ is the set of pure strategies of the second player, and $A = [a_{ij}]$ is a payoff matrix, which says that if the first player uses the strategy x_i, and the second plays according to the strategy y_j, the first one wins (and the second one looses) a_{ij}.*

Definition. *By a* **mixed strategy** *for the first player, whose pure strategies are $X = \{x_1, \dots, x_n\}$, we mean the vector $\mathbf{p} = (p_1, \dots, p_n)$ of probabilities, where p_i is the probability that the player plays according his ith strategy x_i. Analogously, we define a mixed strategy $\mathbf{q} = (q_1, \dots, q_m)$ for the second player.*

Theorem (**Minimax Theorem**). *For every two person zero-sum game there exists a number V, called the* **value of the game**, *such that:*

- *the first player has a mixed strategy $\mathbf{p}$ such that its average win is at least V, no matter what is the strategy of the second player,*
- *the second player has a mixed strategy $\mathbf{q}$ such that its average lost is at most V, no matter what is the strategy of the first player.*

Any mixed strategy who leads to the value of the game is called **optimal** *for this game.*

Fact. *Suppose a game (X, Y, A) contains a* **saddle** *point, i.e. there is an entry a_{ij} of a matrix A such that a_{ij} is both the smallest entry in the ith row and the largest entry in the jth column, then the pure strategies x_i and y_j are optimal for both players and a_{ij} is the value of the game.*

Fact. *If a row i of A is dominated by a row i', i.e. for every j we have $a_{ij} \le a_{i'j}$, then there is an optimal strategy which is not using pure strategy x_i. Consequently, the value of the game (X, Y, A) is the same as the value of the game $(X^{(-i)}, Y, A^{(-i)})$, where $X^{(-i)} = X \setminus \{i\}$, and $A^{(-i)}$ is obtained from A by deleting the ith row. An analogous statement holds when a column j is dominated by a column j'.*

Theorem. *Let (X, Y, A) denote the game in which each player has two strategies and A contains no saddle points. Then to compute the optimal strategy $\mathbf{p} = (p_1, p_2)$ one need to choose p_1 and p_2 in such a way that the average win of the first player is the same for each pure strategy for the second player, i.e. one needs to solve the system of equations*

$$\begin{cases} p_1 a_{11} + p_2 a_{21} &= p_1 a_{12} + p_2 a_{22} \\ p_1 + p_2 &= 1 \end{cases}.$$

In a similar way, the optimal strategy $\mathbf{q} = (q_1, q_2)$ for the second player is a solution to

$$\begin{cases} q_1 a_{11} + p_2 a_{12} &= q_1 a_{21} + q_2 a_{22} \\ q_1 + q_2 &= 1 \end{cases}.$$

Homework assignment for Quiz 11, June 6th.

Problem 1. Zadanie 2. Consider a game with the payoff matrix given by

	L	P
A	2	0
B	-4	1

Find the value of the game. Is the pure strategy A optimal for the first player?

Problem 2. Find the value of a game with the payoff matrix $A = \begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix}$.

Problem 3. Verify that vectors $(2/5, 0, 3/5)$ and $(0, 0, 2/5, 3/5)$ give optimal strategies for the first and the second players respectively in a game with the payoff matrix

$$A = \begin{bmatrix} 5 & 10 & 7 & 1 \\ 2 & 3 & 1 & 3 \\ 2 & 2 & 1 & 5 \end{bmatrix}$$

Problem 4. Rob and Bob play the following game. At the same time each of them rises either one or two hands. If both of them rise two hands, Rob wins 2$, if both of them rise one hand each, Bob wins 1$, otherwise there is a draw (and no payment). Find the value of the game and the optimal strategies for both players.

Problem 5. Two players Odd and Even plays the following game. Each of the players choose a number from the set $\{1, 2\}$. If the sum of the chosen numbers is odd Odd wins, otherwise Even wins. Moreover, the win is always equal to the sum of the two numbers. Find the value of the game and the optimal strategies for both players.

Consider a similar game in which Odd choses a number from the set $\{1, 2\}$ and Even from the set $\{1, 4\}$.