Introduction to Probability Theory
12th week: Markov chains. Ergodic theorem.

Definition. *A Markov chain is* **irreducible** *if for each pair of states $i, j \in S$ there exists a $t \geq 1$ such that $p_{ij}^t > 0$.*

Definition. *The* **period** $d(i)$ *of a state $i \in S$ of a Markov chain is defined as*

$$d(i) = gcd\{t : p_{ii}^t > 0\}.$$

If $d(i) > 1$ we call i **periodic***, otherwise we say that i is* **aperiodic***.*

Remark. If for every t we have $p_{ii}^t = 0$ (and so $d(i)$ is not well defined), we say that the state $i \in S$ is aperiodic.

Fact. *In an irreducible Markov chain all states have the same period.*

Definition. *A Markov chain is* **aperiodic** *if all its states are aperiodic, and it is* **ergodic** *if it is aperiodic and irreducible.*

Theorem 1 (**Ergodic Theorem**). *Each ergodic Markov chain has the unique stationary distribution $\bar{\pi} = (\pi_1, \ldots, \pi_s)$.*

Moreover, for every pair of states $i, j \in S$ we have

$$\lim_{t\to\infty} p_{ij}^t = \pi_j\,,$$

i.e. no matter what is the initial distribution $\bar{p}^0$, for every j we have

$$\lim_{t\to\infty} \bar{p}_j^t = \pi_j.$$

The uniform distribution is always stationary for symmetric Markov chains, namely the following holds.

Theorem 2. *If the transition matrix of a Markov chain (with finite number of states) is symmetric, i.e. for each pair of states we have $p_{ij} = p_{ji}$, then the uniform distribution $\bar{\pi} = (1/|S|, 1/|S|, \ldots, 1/|S|,)$ is a stationary distribution for this chain.*

Note that, unless the chain is ergodic, it can have other stationary distributions as well.

Definition. *A* **random walk** *on a graph $G = (V, E)$ is a Markov chain in whose states are vertices of G and for any two vertices i and j which are adjacent in G we have $p_{ij} = 1/\deg(i)$.*

Homework assignment for Quiz 9, May 23th.

Problem 1. Let us return to our meadow split into the northern and southern parts, and two toads Tail and Head. Now the rules of the game are the following. Each time the coin flip results in the head, Head with probability 1/3 jumps to the other part of the meadow, and with probability 2/3 does nothing. Tail reacts only if there is the tail – in this case with probability 1/2 it jumps over the wall to the other part, and with probability 1/2 it does nothing. Use Ergodic Theorem to estimate the probability that after 125198 flips both toads will sit on the northern parts of the meadow, not knowing what are they starting position. Note that you need to check if the appropriate Markov chain is irregular and aperiodic.

Problem 2. Consider a Markov chain with transition matrix

$$\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{2}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Is this chain irreducible? Is it periodic? Find all its stationary distributions.

Problem 3. The transition matrices for three Markov chains are given by:

$$\text{(a)} \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For each of the chains above:

(a) check if it is irreducible;
(b) find all periodic states (if there are any);
(c) find all stationary distributions.

Problem 4. Consider a random walk on a cycle of length five with vertices w_1, w_2, w_3, w_4, w_5, which starts at w_1.

(a) Are this Markov chain irreducible? Is it periodic?
(b) Find its stationary distribution.
(c) What is $\lim_{t\to\infty} p^t_{w_1 w_3}$?

Problem 5. Consider a random walk on a connected, non-bipartite 4-regular graph G with 15 vertices. Let v and w be two adjacent vertices of G. What is $\lim_{t\to\infty} p^t_{vw}$? Is the limit the same for a pair of non-adjacent vertices?