Introduction to Probability Theory
11th week: Markov chains

Definition. *By a* **Markov chain** *we shall mean a sequence of random variables $X_0, X_1, \ldots$ such that for every $i_0, i_1, \ldots, i_n$ we have*

$$\mathbb{P}\left(X_t = i_t | X_{t-1} = i_{t-1}, X_{t-2} = i_{t-2}, \ldots, X_0 = i_0\right) = \mathbb{P}\left(X_t = i_t | X_{t-1} = i_{t-1}\right) .$$

We shall restict ourselves to the case when all random variables from Markov chain takes the value only from a **finite** set $S \subseteq \mathbb{R}$, and usually take $S = \{1, 2, \ldots, s\}$. The elements of S are the states of this Markov chain. If $X_t = s$ we say that at time t the chain 'is in state s', or 'have the value s', or ' visits i'.

We shall only deal with **homogeneous** Markov chains, i.e. we always assume that for every t and each pairs of states $i, j \in S$

$$\mathbb{P}\left(X_t = j | X_{t-1} = i\right) = \mathbb{P}\left(X_1 = j | X_0 = i\right) .$$

The **t-step transition matrix** $\Pi^{(t)} = [p_{ij}^t]$ consists of the probabilities

$$p_{ij}^t = \mathbb{P}\left(X_t = i | X_0 = j\right) .$$

If $t = 1$ we omit the argument, i.e. we set $p_{ij} = p_{ij}^1$ and $\Pi = \Pi^{(1)}$.

Fact 1. *If $\Pi^{(t)} = [p_{ij}^t]$ then for every i we have*

$$\sum_{j=1}^{s} p_{ij}^t = 1 .$$

Theorem 1. *For every $t \geq 1$ we have*

$$p_{ij}^t = \sum_{k=1}^{s} p_{ik} p_{kj}^{t-1} ,$$

and so

$$\Pi^{(t)} = \Pi^t .$$

Fact 2. *Let $\rho_i^t = \mathbb{P}\left(X_t = i\right)$ and $\bar{p}^t = [\rho_i^t]$. Then*

$$\bar{p}^t = \bar{p}^{t-1}\Pi = \bar{p}^0 \Pi^{(t)} = \bar{p}^0 \Pi^t .$$

Definition. *If $\bar{\pi} = [\pi_i]$ is a vector such that*

$$\bar{\pi}\Pi = \bar{\pi} ,$$

$\sum_{i=1}^{s} \pi_i = 1$ *and* $\pi_i \geq 0$ *for* $i = 1, 2, \ldots, s$, *then* $\bar{\pi}$ *is called a* **stationary distribution** *of a Markov chain with the transition matrix* Π.

Theorem 2. *Each (homogeneous) Markov chain (with finite number of states) has at least one stationary distribution.*

Homework assignment for Quiz 8, May 16th.

Problem 1. Find p_{54}^2, p_{54}^4, and p_{54}^5, for a Markov chain with the following transition matrix

$$\Pi = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 \end{bmatrix} .$$

Problem 2. The transition matrix of a Markov chain is given by

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{2}{3} & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}.$$

(a) Find p^2_{13}.
(b) Represent this Markov chain as an oriented weighted graph on five vertices.
(c) Find p^{100}_{23}.
(d) Find the distribution $\bar{p}^1$, provided we start with the initial distribution $\bar{p}^0 = (\frac{1}{2}, \frac{1}{3}, 0, 0, \frac{1}{6})$.

Problem 3. A frog sits at the side of the stream and each minute with probability $p = \frac{1}{3}$ jumps over it to the other side of the stream, and with probability $q = \frac{2}{3}$ stays where it is. Let X_n denote the position of the frog (one means one side of the stream, two stands for its other side) after n minutes. Thus, (X_n) is a Markov chain with the set of states $S = \{1, 2\}$.

(a) Represent this Markov chain as a weighted oriented graph, whose vertices are states, and arc (and loops) have weights corresponding to probabilites of the transition from one state to another.
(b) Write down the transition matrix Π for this chain.
(c) Suppose that initially the frog at the first side of the stream. What is the probability that after three minutes it will be on the other side? Compute this probability without computing Π^3.
(d) Compute Π^3 and check if p^3_{12} is the same as in (c).
(e) Suppose that initially the frog is on the first side of the river. Compute the distributions $\bar{p}^1$, $\bar{p}^2$, and $\bar{p}^3$.
(f) Suppose now that the initial distribution of a frog is $\bar{p}^0 = (\frac{1}{2}, \frac{1}{2})$, i.e. with equal probability it occupies the first and the second side of the stream. What is the probability that after 3 minutes it will be on the first side?
(g) Find the stationary distribution for this chain.

Problem 4. A meadow is divided by a path into the northern and southern parts. Two toads, named Tail and Head, are sitting on the opposite sides of the path. From time to time they flip a coin. In case of the head Head jumps over the path to the other part of the meadow, in case of the tail Tail changes its position.

(a) Model this situation by a Markov chain with four states.
(b) Write down the transition matrix for this chain.
(c) Let us assume that at the beginning of the game both toads are at the northern part of the meadow. What is the probability that after five flips both of them will be at the southern part of the meadow?
(d) Suppose that, as before, both toads are initially at the northern part of the meadow. What is the probability that after 12345 flips they both will be at the southern part?
(e) Find the stationary distribution for this chain.

Problems we shall do in class on May 16th

Problem 1. The transition matrix of a Markov chain is given by

$$\Pi = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and the initial distribution for this chain is $\bar{p}^0 = (0, \frac{1}{2}, 0, \frac{1}{2})$. Find $\bar{p}^1$ and $\bar{p}^2$.

Compute $\Pi^{(2)} = \Pi^2$ and use this result to find $\bar{p}^2$ without computing $\bar{p}^1$.

Gambler's Ruin Consider a Markov chain with transition matrix

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ p & 0 & 1-p & 0 \\ 0 & p & 0 & 1-p \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $p \in (0,1)$. For every $t \geq 2$ find p_{24}^t and compute $\lim_{t\to\infty} p_{24}^t$.

Problem 2. Let us play game with two urns. The first urn contains 3 white balls, the second 3 black balls. In the n-th round of the game we pick up one ball from each of the urns and swap them. Let X_n denote the number of white balls in the first urn after n rounds. Find the transition matrix for the Markov chain $(X_n)_{n=0}^{\infty}$.

Problem 3. Three fireflies sit at the vertices of a triangle and each of them can emanate both red and green light. Every second each of them changes its color according to the following rule. If it sees that two other fireflies are of the same color in the next second it will change into their color, otherwise with probability 1/2 it will be red and with probability 1/2 green. Construct a Markov chain corresponding to this phenomena and write down its transition matrix provided

a) we distinguish fireflies (which means we distinguish vertices of the triangle they sit at);
b) we are interested only in how many fireflies are red and how many of them are green;
c) we distinguish neither fireflies nor colors.

Find the stationary distribution for c) and try to guess the stationary distributions for a) i b). Is it true that in each for these cases there is only one stationary distribution?

Problem 4. Let us assume that in the previous problem fireflies behave in a slightly different way. When a firefly sees that two other ones are of the same color then with probability 2/3 changes its color into that color, and with probability 2/3, it changes to the other color. Find three transition matrices corresponding to the cases a), b), c), in the previous problem. In case of c) find the stationary distribution.
